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Vacuum Energy and Casimir Force in a Presence of Skin-depth Dependent Boundary Condition

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The vacuum energy-momentum tensor (EMT) and the vacuum energy corresponding to massive scalar field on $\Re_t \times [0,l] \times \Re^{D-2}$ space-time with boundary condition involving a dimensional parameter (δ) are found. The dependent on the cavity size l Casimir energy $\widetilde{E}_C$ is a uniquely determinable function of mass m, size l and "skin-depth" δ. This energy includes the "bulk" and the surface (potential energy) contributions. The latter dominates when $l \sim \delta$. Taking the surface potential energy into account is crucial for the coincidence between the derivative $-\partial\widetilde{E}_C/\partial l$ and the ll-component of the vacuum EMT. Casimir energy $\widetilde{E}_C$ and the bulk contribution to it are interconnected through Legendre transformation, in which the quantity δ^{-1} is conjugate to the vacuum surface energy multiplied by δ. The surface singularities of the vacuum EMT do not depend on l and, for even D, $\delta = 0$ or ∞, possess finite interpretation. The corresponding vacuum energy is finite and retains known analytical dependence on the dimension D.

1. Introduction

The determination of vacuum energy of fields confined to finite volumes is the basic part in the calculations of Casimir force as well as of some bag characteristics of hadrons [1, 2]. At the same time the renormalization approach appears to be not unique: the counterterms necessary to make the vacuum energy finite can depend on the cavity size so that the regularization dependence of Casimir energy appears [3, 4]. Concrete calculations for the scalar case and Dirichlet or Neumann boundary condition on a spherical boundary have been performed in the recent papers [5, 6, 7]. General consideration [8] shows however that at least for these boundary conditions the surface singularities of the vacuum EMT (and, hence, the aforementioned counterterms) can not depend on the size of the cavity. The detection of such dependences in the papers listed above is a consequence of a coincidence between the local (curvature) and global (size) parameters determining the spherical boundary.

We consider the vacuum characteristics of massive scalar field defined on the domain $\Re_t \times [0,l] \times \Re^{D-2}$. In the absence of curvature we introduce "skin-depth" parameter δ [9] in boundary condition [2)]

$$\partial_1\varphi(t,0,\mathbf{x}_\perp) = \frac{1}{\delta}\varphi(t,0,\mathbf{x}_\perp), \quad \varphi(t,l,\mathbf{x}_\perp) = 0 \tag{1}$$

This boundary condition implies the presence of the surface potential energy [10]

$$\frac{1}{2\delta}\int d^{D-2}\mathbf{x}_\perp\, \varphi^2(t,0,\mathbf{x}_\perp) \tag{2}$$

in the total hamiltonian of the model. The latter is, as usual, space integral of 00-component of the energy-momentum tensor

$$\widetilde{T}_{\mu\nu} = \frac{1}{2}\left[\partial_\mu\varphi, \partial_\nu\varphi\right]_+ - \frac{1}{2}g_{\mu\nu}(\partial\varphi)^2 + \frac{1}{2}g_{\mu\nu}m^2\varphi^2 + \xi\partial^\lambda(g_{\mu\nu}\left[\varphi, \partial_\lambda\varphi\right]_+ - g_{\mu\lambda}\left[\varphi, \partial_\nu\varphi\right]_+) \tag{3}$$

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2) We use the system of units where $\hbar = c = 1$. The signature of metric is $(+,-,-,\ldots)$. $x^\mu = (t, x_1, \mathbf{x}_\perp)$. Brackets $[,]_+$ mean anticommutator.

where the last divergence term should be taken with coefficient $\xi = 1/4$. This is the only value of ξ guaranteeing the diagonality of hamiltonian and its conservation [10].

2. Energy-momentum tensor and energy of vacuum

The normalized solutions of Klein-Gordon-Fock equation have the form:

$$\widetilde{\varphi}_k(x) = (2\omega)^{-\frac{1}{2}}(2\pi)^{1-\frac{D}{2}} \exp\left[-i\omega t + i\mathbf{q}\mathbf{x}_\perp\right]\psi_k(x_1), \tag{4}$$

where $\omega = \sqrt{m^2 + \mathbf{q^2} + k^2}$, and the descrete set of functions

$$\psi_k(x) = N_k \sin k(x-l), \quad N_k = \left[\frac{l}{2}\left(1 - \frac{\sin 2kl}{2kl}\right)\right]^{-1/2} \tag{5}$$

represents the resonator modes. Wave number $k = z/\delta$ $\quad(\widetilde{l} \equiv l/\delta)$ should be determined from the spectral equation

$$\Delta(z) \equiv z^{-1}\,\mathrm{tg}\, z\widetilde{l} + 1 = 0, \tag{6}$$

following from boundary condition (1). The basis of functions (4) is used to define creation-annihilation operators and vacuum state in a standard manner [1, 11] [3)].

With the help of Cauchy theorem applied to meromorphic function $\Delta(z)$, it is possible to transform the sums over transcendential roots of eq.(6) into corresponding integrals. The scheme of calculation fits in the formula

$$\sum_{z_{n\delta}>0} \frac{f(z_{n\delta})}{1 - \dfrac{\sin 2z_{n\delta}\widetilde{l}}{2z_{n\delta}\widetilde{l}}} = \frac{\widetilde{l}}{\pi}\int\limits_0^\infty f(z)\,dz - \frac{\widetilde{l}}{2\pi i}\int\limits_0^\infty \frac{(1-t)[f(it) - f(-it)]}{\mathrm{sh}\, t\widetilde{l} + t\,\mathrm{ch}\, t\widetilde{l}}\, e^{-t\widetilde{l}}\, dt, \tag{7}$$

where $f(z)$ is a function analitical in the right half-plane, and $z_{n\delta}\,(n = 1, 2, \cdots)$ are the (real) roots of eq.(6). With the use of eq.(7) one could attach the integral representation to the un-renormalized vacuum EMT. The renormalization is reduced to the subtraction of Minkowskian space contribution (D-regularization [12] supposed):

$$\langle T_{\mu\nu}\rangle_M = \frac{\Gamma(\frac{1-D}{2})}{\sqrt{\pi}(4\pi)^{D/2}}\int_0^\infty dk\; M^{D-3}\,\mathrm{diag}\begin{bmatrix} -M^2 \\ k^2(1-D) \\ M^2 \end{bmatrix}, \tag{8}$$

$M \equiv (m^2 + k^2)^{1/2}$. The final answer i.e. renormalized EMT of vacuum, looks like

$$\langle \widetilde{T}_{\mu\nu}\rangle_{\delta,l} = \frac{\mathrm{K}_D}{\delta}\int\limits_\mu^\infty dt(t^2-\mu^2)^{\frac{D-1}{2}}\left\{\frac{(1-t)e^{-t\widetilde{l}}}{\mathrm{sh}\, t\widetilde{l} + t\,\mathrm{ch}\, t\widetilde{l}}\,\mathrm{diag}\begin{bmatrix} \mathrm{ch}\, 2tx' - 1 \\ \dfrac{t^2(1-D)}{t^2-\mu^2} \\ 1 - \mathrm{ch}\, 2tx' \end{bmatrix} + e^{-2tx'}\,\mathrm{diag}\begin{bmatrix} 1 \\ 0 \\ -I \end{bmatrix}\right\}. \tag{9}$$

$(D-2)$-dimensional diagonal of $\langle \widetilde{T}_{\mu\nu}\rangle_{\delta,l}$ corresponding to $\mathbf{x}_\perp$ contains expressions of the same type. The following notations were used in eq.(9):

$$\mathrm{K}_D = \frac{\delta^{1-D}}{2(4\pi)^{\frac{D-1}{2}}\Gamma\left(\frac{1+D}{2}\right)}, \; \mu = m\delta, \; x' = (l - x_1)/\delta. \tag{10}$$

[3)] It should be noted, however, that canonical quantization of the system (3) for $\xi = 1/4$ as a system with a higher derivatives, should account for its degeneracy [10].

The energy of vacuum could be determined directly, without resort to eq.(9), as a sum of half-frequences interpreted e.g. with the help of zeta-regularization method [11]. At the same time, Green function method [1, 8, 11] used to obtain vacuum energy should rely on the tensor $\widetilde{T}_{\mu\nu}$ (3) [10], but not its first three terms

$$T_{\mu\nu} = \frac{1}{2}\left[\partial_\mu\varphi, \partial_\nu\varphi\right]_+ - \frac{1}{2}g_{\mu\nu}(\partial\varphi)^2 + \frac{1}{2}g_{\mu\nu}m^2\varphi^2. \tag{11}$$

11-component of tensor (3) gives Casimir pressure $P = \langle\widetilde{T}_{11}\rangle_{\delta,l}$ which, because of translational symmetry, coincides with $\langle T_{11}\rangle_{\delta,l}$ (i.e. the divergence term in (3) does not affect the pressure). The role of divergence term is evident from the fact that value $\xi = 1/4$ is the only one at which equality

$$-\frac{\partial\widetilde{E}_{vac}}{\partial l} = \langle\widetilde{T}_{11}\rangle_{\delta,l} \tag{12}$$

holds. Notice that conservation of the total energy [10] and equality (12) both rely on the value $\xi = 1/4$.

In proving the eq.(12) it is essential that vacuum energy per unit area of the boundary

$$\widetilde{E}_{vac} = \int_{x_0}^{l-y_0} dx_1\, \langle\widetilde{T}_{00}\rangle_{\delta,l} = \widetilde{E}_C(l,\delta) + \widetilde{E}_{w1}(x_0,\delta) + \widetilde{E}_{w2}(y_0,0) + \cdots, \tag{13}$$

where dots denote terms vanishing at $x_0, y_0 \to 0$ as well as at $l \to \infty$. Only finite part of $\widetilde{E}_{vac}$, i.e. Casimir energy

$$\widetilde{E}_C(l,\delta) = \mathrm{K}_D \int\limits_\mu^\infty dt\, \frac{(t^2-\mu^2)^{\frac{D-1}{2}} e^{-t\tilde{l}}}{\operatorname{sh} t\tilde{l} + t\operatorname{ch} t\tilde{l}} \left[\tilde{l}(t-1) - \frac{1}{t+1}\right], \tag{14}$$

depends on the size of the domain and vanishes when $l \to \infty$. Boundary divergences are present in r.h.s. of eq.(13) in the form of the energies of the walls, $\widetilde{E}_{w1,2}$. For example,

$$\widetilde{E}_{w1}(x_0,\delta) = \mathrm{K}_D \mu^{D-1} \int\limits_1^\infty \frac{d\xi}{2\xi}\, (\xi^2-1)^{\frac{D-1}{2}}\, \frac{1-\mu\xi}{1+\mu\xi}\, e^{-2mx_0\xi}. \tag{15}$$

The specific representation of energies $\widetilde{E}_{w1,2}$ depends on regularization employed, but Casimir energy is independent of the latter. Notice that the terms discarded in the r.h.s. of eq.(13) have the property of "double vanishing" (with respect to limits of x_0, y_0 and l) so that energies $\widetilde{E}_{w1,2}$ of the walls are determined uniquely *within* the given regularization scheme.

By a complete analogy with $\widetilde{E}_{vac}$, one can find its bulk (E_{vac}) and surface ($\Pi_{vac} = \widetilde{E}_{vac} - E_{vac}$) components, the first dertermined with the help of density T_{00} (11). For either of these components the expansion of the form (13) exists giving rize to $E_C(l,\delta)$ and $\Pi_C(l,\delta)$ correspondingly. For example,

$$\Pi_C(l,\delta) = \mathrm{K}_D(1-D) \int\limits_\mu^\infty dt\, \frac{t^2(t^2-\mu^2)^{\frac{D-3}{2}} e^{-t\tilde{l}}}{(t+1)(\operatorname{sh} t\tilde{l} + t\operatorname{ch} t\tilde{l})}, \tag{16}$$

and $E_C = \widetilde{E}_C - \Pi_C$. Since $\Pi_C(l,\delta) \neq 0$, derivative $-\partial E_C/\partial l$ does not coincide with the Casimir pressure (12). Nevertheless,

$$\delta \cdot \frac{\partial\widetilde{E}_C}{\partial\delta} = -\Pi_C, \tag{17}$$

and, with the notations $\lambda = \delta^{-1}, f = \Pi_C/\lambda$, one finds that $\widetilde{E}_C$ and E_C are interconnected through Legendre transformation:

$$\widetilde{E}_C(l,\lambda) = E_C(l, f(l,\lambda)) + \lambda f(l,\lambda), \tag{18}$$

so that

$$\left(\frac{\partial E_C}{\partial l}\right)_f = \left(\frac{\partial \widetilde{E}_C}{\partial l}\right)_\lambda, \text{ and } \left(\frac{\partial E_C}{\partial f}\right)_l = -\lambda. \tag{19}$$

Now energy E_C determines the Casimir pressure but under specific condition of constancy of the quantity f being one half the "Casimir" part of $\langle\varphi(0)^2\rangle$, see (2).

3. Asymptotic properties

For the purpose of comparison with electrodynamics, here we consider a massless case. The behaviour of integral (14) in Dirichlet ($\delta \ll l$) and Neumann ($\delta \gg l$) regimes is displayed by the following expansions ($m = 0$):

$$\widetilde{E}_C(l,\delta) = \mathrm{K}_D \, \frac{\zeta(D)\Gamma(D)}{(2\tilde{l})^{D-1}} \left[-1 + C^1_{D-1}\,\tilde{l}^{-1} - C^2_D\,\tilde{l}^{-2} + C^3_{D+1}\left(1 + \frac{\zeta(D+2)}{2\zeta(D)}\right)\tilde{l}^{-3}\right], \tag{20}$$

$$\widetilde{E}_C(l,\delta) = \mathrm{K}_D \tilde{l}^{-D+1} \begin{cases} A_D - (D-1)A_{D-2}\tilde{l}, & D = 3,4,\ldots, \\ A_2 + \tilde{l}\ln(4\gamma_E\tilde{l}/\pi) - \tilde{l}, & D = 2. \end{cases} \tag{21}$$

Here C^m_n denotes binomial coefficient, $A_D = 2^{1-D}(1 - 2^{1-D})\Gamma(D)\zeta(D)$, $\ln\gamma_E = 0.577$, and eq.(21) contains the leading corrections only. The expansion of pressure $\langle\widetilde{T}_{11}\rangle_{\delta,l}$ corresponding to (20) has the form ($\delta \ll l$):

$$\langle\widetilde{T}_{11}\rangle_{\delta,l} = \frac{(D-1)\Gamma(D)\zeta(D)}{(4\pi)^{\frac{D-1}{2}}\Gamma\left(\frac{1+D}{2}\right)}(2l)^{-D}\times$$
$$\times\left[-1 + C^1_D\tilde{l}^{-1} - C^2_{D+1}\tilde{l}^{-2} + C^3_{D+2}\left(1 + \frac{\zeta(D+2)}{2\zeta(D)}\right)\tilde{l}^{-3} + \ldots\right], \tag{22}$$

and for Neumann regime (21)

$$\langle\widetilde{T}_{11}\rangle_{\delta,l} = \frac{(D-1)\,l^{-D}}{2(4\pi)^{\frac{D-1}{2}}\Gamma\left(\frac{D+1}{2}\right)}\left[A_D - (D-2)A_{D-2}\,\tilde{l} + \ldots\right]. \tag{23}$$

The four terms in the square brackets of eq.(22) (term with ζ-functions excluding) could be obtained through the Taylor series for $(l+\delta)^{-D}$. Thus, eq.(22) demonstrates the role of δ as penetration depth [9]. Unlike Dirichlet case (22), the correction term in the brackets of eq.(23) emerges due to surface energy Π_C only. Numerical analysis of the formulas (14) and (16) taken at $m = 0$ shows the dominance of the surface contribution Π_C over the bulk one (E_C) in the region $l \sim (1 \div 4)\delta$. The difference in signs between those quantities is responsible for the shift of the position of the minimum from $l \sim 4.5\delta$ (for E_C) to $l \sim 1.8\delta$ (for $\widetilde{E}_C$). At that time, the qualitative behaviours of energies E_C and $\widetilde{E}_C$ as functions of l are alike showing a typical van-der-vaals character.

At $D = 4$ formula (22) takes the form

$$\langle \widetilde{T}_{11} \rangle_{\delta,l} = \frac{\pi^2}{480 l^4} \left[-1 + 4\frac{\delta}{l} - 10\frac{\delta^2}{l^2} + 20\left(1 + \frac{9\pi^2}{185}\right)\frac{\delta^3}{l^3} + \cdots \right], \tag{24}$$

and should be compared with the corresponding expression in the case of electromagnetic field confined to the region between impedance walls (δ- skin-depth parameter):

$$P = \frac{\pi^2}{240 l^4} \left[-1 + \frac{16}{3}\frac{\delta}{l} - 24\frac{\delta^2}{l^2} + \frac{640}{7}\left(1 + \frac{9\pi^2}{740}\right)\frac{\delta^3}{l^3} + ... \right]. \tag{25}$$

Formula (25) except the last term $\sim \delta^3/l^3$ in the square brackets, was extracted from [1, 9]. The corresponding coefficients of expansions (25) and (24) are related to each other in ratios 1, 1.33, 2.4, 3.46, hence showing the growing effect of spin when going deeper into the boundary.

4. Interpretation of surface singularities at $\delta = 0$ and ∞

The surface singularities of the vacuum EMT is a stumbling-block problem for any field theory expected to establish the connection between its local properties and observational predictions [8]. It is shown below that the method of dimensional regularization could be applied not only to interpret the singular sums like $E_{vac} = \sum\limits_{\nu} \frac{1}{2}\omega_\nu$, but gives a reasonable finite answer for the energy density of vacuum as well. The method works for even $D, \delta = +0$ [4)] or ∞ and leads to an energy expression consistent with the one of ζ-regularization method.

Boundary divergences are represented in r.h.s. of eq.(13) by the energies of the walls ($\widetilde{E}_{w1,2}$). Correspondingly, we consider 00-component of a tensor (9) in the limit $l \to \infty$, $\delta = +0$ (alternative case $\delta = \infty$ differs from (26) only in sign):

$$\langle \widetilde{T}_{00} \rangle_{0,\infty} = \left(\frac{m^2}{2\pi}\right)^{\frac{D}{2}} \begin{cases} \rho^{-D/2}\, K_{D/2}(\rho), & 0 < \rho \lesssim 1,\ x/l \to 0; \\ (\rho')^{-D/2}\, K_{D/2}(\rho'), & 0 < \rho' \lesssim 1,\ x'/l \to 0 \end{cases} \tag{26a, b}$$

($\rho = 2mx, \rho' = 2mx' = 2m(l - x)$). Exploiting recurrence relations between modified Bessel functions $K_\nu(\rho)$ [13], let us transform energy density (26a):

$$\langle \widetilde{T}_{00} \rangle_{0,\infty} \to \langle \widetilde{T}_{00} \rangle_{0,\infty} - \frac{d}{d\rho} f_D(\rho) = \left(\frac{m^2}{2\pi}\right)^{\frac{D}{2}} \frac{\rho^{1-\epsilon}\, K_{\epsilon-1}(\rho)}{(1-D)(3-D)\cdots(1-2\epsilon)}. \tag{27}$$

Here $\epsilon \equiv \frac{D}{2} - n \to 0$, $n = 1, 2, \ldots$, and function

$$f_D(\rho) = \left(\frac{m^2}{2\pi}\right)^{\frac{D}{2}} \left[\frac{\rho^{1-\frac{D}{2}} K_{\frac{D}{2}}(\rho)}{1-D} + \frac{\rho^{2-\frac{D}{2}} K_{\frac{D}{2}-1}(\rho)}{(1-D)(3-D)} + \ldots + \frac{\rho^{1-\epsilon} K_\epsilon(\rho)}{(1-D)(3-D)\cdots(1-2\epsilon)} \right] \tag{28}$$

possesses following properties: $i)$ for $\Re e D < 0$ ($\Re e\ \epsilon < -n$)

$$f_D(0) = f_D(\infty) = 0, \tag{29}$$

[4)] The singular nature of Dirichlet limit (contrary to Neumann one) is explained in [10].

so that divergence addition in the l.h.s. of eq.(27) does not affect the vacuum energy of the wall [6, 10, 14]

$$\widetilde{E}_{w1}(0,0)=\left(\frac{m^2}{2\pi}\right)^{\frac{D}{2}}\int_0^\infty dx\,\rho^{-D/2}\,K_{D/2}(\rho)=\frac{m^{D-1}\Gamma\left(\frac{1-D}{2}\right)}{8(4\pi)^{\frac{D-1}{2}}}. \tag{30}$$

(the vacuum energy of Neumann wall $\widetilde{E}_{w1}(0,\infty)=-\widetilde{E}_{w1}(0,0)=-E_{w1}(0,0)$);

ii) a term $f'_D(\rho)$ in the l.h.s. of eq.(27) taken at "physical" values of $D=2,4,6,\cdots$, acts as a counterterm eliminating all nonintegrable singularities of density (26a);

iii) this counterterm preserves an exponential decreasing behaviour of the modified density (27) at $x>\hbar/mc$. The latter is not still uniquely defined: one can add to $f_D(\rho)$ any regular function having the property (29) (see [8, 11]).

Thus, at $\delta=+0$ the total energy of vacuum includes Casimir part [14]

$$\widetilde{E}_C(l,0)=-\frac{m^D l}{(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D+1}{2})}\int_1^\infty d\xi\,\frac{(\xi^2-1)^{\frac{D-1}{2}}}{e^{2\xi ml}-1} \tag{31}$$

and doubled topological (according to [1]) energy (30) caused by the presence of the edge of the manifold. At $\delta=\infty$ the total energy of vacuum reduces to $\widetilde{E}_C(l,\infty)$ because the energies of Neumann ($x=0$) and Dirichlet ($x=l$) walls cancell out. Energy $\widetilde{E}_C(l,\infty)=E_C(l,\infty)$. The integral representation for it can be obtained from (31) by substitution "+" for "−" in the denominator of the integrand and by multiplying the whole expression by (-1).

Dimensional regularization method does not result in satisfactory expressions for the vacuum energy when $0<\delta<\infty$. As it may be seen from eq.(15) (taken at $x_0=0$), for even D the corrections of order $\mu,\mu^2,\cdots$ to the energy (30) cannot be interpreted in the way like (30). On the other hand, for odd $D\geq 3$ the energy (30) is infinite. Now, the depending on μ "corrections" find finite interpretation, but being summed up, they lead to logarithmic divergence in the Neumann limit $\mu\to\infty$.

The analytical structure of the surface singularities of the vacuum EMT (26) with respect to dimension D is just that property which makes dimensional regularization applicable to define finite density (27). In view of the absence of translational symmetry it seems quite natural for density like (27) to exist. At that time finite renormalization of the type suggested in [6, 14] eliminates the energy of the walls from $\widetilde{E}_{vac}=\widetilde{E}_C(l,0)+2\widetilde{E}_w$. This procedure needs in explaination how to interpret the expressions like (27). It should be mentioned in addition that the energy density (27) (but not (26)) vanishes when $m=0$. This disappearance was assumed to be characteristic of conformally invariant models [11, 15].

5. **Conclusion**

Below some comments on literature related to the present topic are given. The massless case for the scalar model at hand (dimension $D=2,3$) was considered in works [9, 16] and formulas consistent with (9) ($D=3, m=0$, [16]) and (14) ($D=2, m=0$, [9]) were derived. The work [16] exploits an idea [17] of replacement of the boundary with a singular non-locally regularized potential, that occured to be equivalent to taking potential energy (2) into account.

Formula (2.18) from [14] corresponds to our formula (31). Doubled energy (30) appears in [14] as well, but it was not associated there with the vacuum energy of half-space. Our formula

(9) at $\delta = +0$ (or, equivalently, (31) and (12)) gives the expression

$$\langle \tilde{T}_{11} \rangle_{0,l} = \frac{m^D (1-D)}{(4\pi)^{\frac{D-1}{2}} \Gamma\left(\frac{1+D}{2}\right)} \int\limits_1^\infty \frac{\xi^2 (\xi^2 - 1)^{\frac{D-3}{2}}}{e^{2ml\xi} - 1} \, d\xi \Bigg|_{ml \gg 1} = \\ = -\frac{m^D}{(4\pi m l)^{\frac{D-1}{2}}} \, e^{-2ml} \left[1 + \frac{(D-1)(D+5)}{16ml} + \cdots \right], \tag{32}$$

which does not coincide with formula (2.13) from [18]. The latter, being taken at zero temperature, includes physically unacceptable dependence of pressure on (arbitrary) renormalization parameter. Such dependence, as it was noted above, is characteristic of curved boundary under the condition of coincidence between the parameters determining its curvature and size.

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